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Characterization Theorems for the Approximation by a Family of Operators

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The intention of this paper is to study a family of positive linear approximation operators relating to most of the well known Bernstein-type operators. These operators depend on a parameter. We give some characterization theorems to show that the operators corresponding to different parameters can be quite different. The direct and converse results make use of the Ditzian–Totik modulus of smoothness.

1. INTRODUCTION AND BACKGROUND

We will study some local and global characterization results for a family of Bernstein type approximation processes P_n on $I := [0, 1]$ defined as

$$P_n(f; x) := \sum_{k=0}^{n} p_{n,k}(x)\, T_{\alpha,k}(f), \tag{1.1}$$

with the positive linear functionals $T_{\alpha,k}\colon C(I) \to \mathbb{R}$ for $k = 0, ..., n$

$$T_{\alpha,k}(f) := \frac{\int_0^1 f(t)\, t^{ck+a}(1-t)^{c(n-k)+b}\, dt}{B(ck+a+1, c(n-k)+b+1)} \qquad a, b > -1 \tag{1.2}$$

and $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in I$, $n \in \mathbb{N}$.

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Here we will use for c the special sequence in n and α, with $n \in \mathbb{N}$, $0 \leqslant \alpha < \infty$, defined as

$$c := c_n := [n^\alpha], \tag{1.3}$$

the integer parts. This family covers among other things the classical Bernstein operators, the Bernstein–Durrmeyer–Lupaş operators (Durrmeyer [10], Lupaş [14]) and the de la Vallée–Poussin type operators (Lupaş and Mache [15]).

It turns out that the approximation properties of some operators in this family P_n are somewhat similar.

We can remark that a generalization of this family was introduced by D. H. Mache [17] in the form

$$D_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 f(\alpha_n t + (1-\alpha_n)(k/n))\, t^{a_k}(1-t)^{b_k}\, dt}{\int_0^1 t^{a_k}(1-t)^{b_k}\, dt}, \tag{1.4}$$

where $\alpha_n \in I$ and $f \in L_{1,\omega}(I)$.

As usual $L_{1,\omega}(I)$ denotes the set of all measurable functions f on I for which the corresponding norm $\|f\|_{1,\omega} := \int_0^1 |f(t)|\, \omega(t)\, dt$ is finite with the weight ω, defined as $\omega(t) := \omega_{a,b}(t) = t^a(1-t)^b$, $a := \inf_{0 \leqslant k \leqslant n,\, 0 \leqslant n < \infty} a_k$, $b := \inf_{0 \leqslant k \leqslant n,\, 0 \leqslant n < \infty} b_k$, where $a_k, b_k > -1$ depend upon $n \in \mathbb{N}$ and $k = 0, 1, ..., n$.

This polynomial sequence $(D_n)_{n \in \mathbb{N}}$ includes a number of well-known positive linear approximation operators:

- For $\alpha_n = 1/(n+1)$ and $a_k = b_k = 0$, $k = 0, 1, ..., n$, we obtain the classical **Bernstein–Kantorovič operators** K_n,

- for $\alpha_n = 0$ and also $a_k = b_k = 0$, $k = 0, 1, ..., n$, the classical **Bernstein operators** B_n,

- for $\alpha_n = 1$ and $a_k = k$, $b_k = n - k$, $k = 0, 1, ..., n$, the **Bernstein–Durrmeyer–Lupaş operators** M_n (cf. Durrmeyer [10] and Lupaş [14]) defined by

$$M_n(f; x) := (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t)\, f(t)\, dt,$$

$$x \in [0, 1], \qquad n \in \mathbb{N}, \qquad f \in L_1[0, 1].$$

As a generalization, Berens and Xu proved in the interesting paper [4] a number of nice properties for $a_0 = \cdots = a_n =: a > -1$ and $b_0 = \cdots = b_n =: b > -1$:

- for $\alpha_n = 1$ and $a_k = k + a$, $b_k = n - k + b$, $k = 0, 1, ..., n$, the **operators of Păltănea** M_n^{ab} (Păltănea [21]), defined by

$$M_n^{ab}(f; x) := \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 f(t)\, p_{n,k}(t)\, t^a(1-t)^b\, dt}{\int_0^1 p_{n,k}(t)\, t^a(1-t)^b\, dt},$$
$$x \in [0, 1], \qquad n \in \mathbb{N}, \qquad f \in L_{1,\omega}[0, 1].$$

In the case $a = b = -\frac{1}{2}$, one researched in (see Lupaş and Mache [15]) the de la Vallée–Poussin-type operators V_n,

- for $\alpha_n = 1/(n+1)$ and $a_0 = \cdots = a_n =: a > -1$, $b_0 = \cdots = b_n =: b > -1$ we get a generalized form of the Bernstein–Kantorovič operators (cf. Mache [17]), namely

$$\mathscr{K}_n(f; x) := \sum_{k=0}^{n} p_{n,k}(x) \frac{(\int_0^1 f(1/(n+1)\, t + (k/(n+1)))\, t^a(1-t)^b\, dt)}{B(a+1; b+1)}, \tag{1.5}$$

where $x \in I$, $n \in \mathbb{N}$, $f \in L_{1,\omega}(I)$, and $B(u, v)$ is the beta function for the parameters u and v.

The purpose of this paper is to derive some common and different parts of these known and also somewhat new operators.

But first let us describe in a short way the connection lines to the above operators, for which we will present in the following section the local and global results.

A function h: $[a, b] \to \mathbb{R}$ is called convex, if for all $x_1, x_2 \in [a, b]$ and all $\alpha_n \in I$, Jensen's inequality

$$h(\alpha_n x_1 + (1-\alpha_n)\, x_2) \leqslant \alpha_n h(x_1) + (1-\alpha_n)\, h(x_2)$$

holds. For $x_1 = t$ and $x_2 = k/n$ one obtains for every convex function h

$$D_n(h; x) \leqslant \alpha_n Q_n(h; x) + (1-\alpha_n)\, B_n(h; x), \tag{1.6}$$

where P_n is defined for $f \in L_{1,\omega}[0, 1]$ by

$$Q_n(f; x) := \sum_{k=0}^{n} p_{n,k}(x) \frac{\int_0^1 f(t)\, t^{a_k}(1-t)^{b_k}\, dt}{\int_0^1 t^{a_k}(1-t)^{b_k}\, dt}. \tag{1.7}$$

Here we can say that $D_n h - \alpha_n Q_n h \leqslant (1-\alpha_n)\, B_n h$ is like a disturbance—or interference—operator.

2. Main Results

Now we will present the direct and converse approximation results for the above-mentioned method P_n in (1.1).

To formulate the results, let $(0<h<\frac{1}{2})$

$$\Delta_h^2 f(x) := f(x+h) - 2f(x) + f(x-h), \qquad x \in [h, 1-h] \tag{2.1}$$

$$\omega_2(f;\delta) := \sup_{0<h\leqslant\delta} \sup_{x\in[h,1-h]} |\Delta_h^2 f(x)|, \tag{2.2}$$

$$\mathrm{Lip}^*\beta := \{f \in C(I);\, \omega_2(f;\delta) = \mathcal{O}(\delta^\beta),\, \delta \to 0_+\}, \qquad 0<\beta<2. \tag{2.3}$$

THEOREM 2.1. *Let $(P_n)_{n\in\mathbb{N}}$ be defined as in* (1.1), $f \in C(I)$, *and* $0 \leqslant \alpha < 1$. *If*

- (*Case* 1) $0<\beta<1$ *for* $\alpha=0$, *or*
- (*Case* 2) $0<\beta\leqslant\alpha+1$ *for* $0<\alpha<1$,

then

$$|f(x) - P_n(f;x)| \leqslant C\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\beta/2}, \qquad x \in I \tag{2.4}$$

if and only if

$$\omega_2(f;t) = \mathcal{O}(t^\beta). \tag{2.5}$$

For $\alpha+1<\beta<2$ the above equivalence does not hold.

At this point we mention that in this paper the constant C denotes a positive constant which can be different at each occurrence.

Remark 2.2. The case $\alpha=0$, $\beta=1$ is impossible which can be seen from the example $f(x)=x\ln x$ (Zhou [24]).

THEOREM 2.3. *Let $(P_n)_{n\in\mathbb{N}}$ be defined as in* (1.1), $f \in C(I)$, *and* $\alpha \geqslant 1$. *Then the same equivalence as in Theorem* 2.1 *is true for* $0<\beta<2$.

The proof of this theorem is the same as that of Theorem 2.1 and will be omitted.

Thus, there exists no "cut off" for $\alpha \geqslant 1$ while the "cut off" is in $1+\alpha$ for $0 \leqslant \alpha < 1$. Therefore, we find that the operators P_n for $\alpha \geqslant 1$ are very nice, which can be seen from the following surprising three results (Theorem 2.4, Corollary 2.5, and Theorem 2.6).

THEOREM 2.4. *Let $(P_n)_{n\in\mathbb{N}}$ be defined as in* (1.1), $f \in C(I)$, *and* $\alpha \geqslant 1$. *Then*

$$\|f - P_n f\|_\infty \leqslant C\left(\omega_\varphi^2\left(f;\frac{1}{\sqrt{n}}\right)_\infty + n^{-1-\alpha}\|f\|_\infty\right), \qquad n \in \mathbb{N}, \tag{2.6}$$

where the constant C is independent of n and α, and $\omega_\varphi^2(f;\cdot)_\infty$ denotes the second order Ditzian–Totik modulus of smoothness of f (cf. Ditzian and Totik [9]) especially for the sup-norm and the step-weight $\varphi=\sqrt{x(1-x)}$.

COROLLARY 2.5. *By letting* $\alpha \to \infty$ *in* (2.6) *we have a well-known result for the classical Bernstein operators*

$$\|f - B_n f\|_\infty \leqslant C\omega_\varphi^2\left(f; \frac{1}{\sqrt{n}}\right)_\infty, \qquad n \in \mathbb{N}, \tag{2.7}$$

where the constant C *is independent of* n.

Here one can see that the approximation properties of the above operators are closely related to the smoothness behaviour of the function f they approximate. Without claim of completeness we mention the following interesting papers, which deal with well known Bernstein type operators and their approximation properties [4, 6–8, 22].

THEOREM 2.6. *Let* $(P_n)_{n\in\mathbb{N}}$ *be defined as in* (1.1), $f \in C(I)$, *and* $\alpha \geqslant 1$. *Then for* $0 < \beta < 1$,

$$\|f - P_n f\|_\infty = \mathcal{O}(n^{-\beta}) \tag{2.8}$$

if and only if

$$\omega_\varphi^2(f; t)_\infty = \mathcal{O}(t^{2\beta}). \tag{2.9}$$

Remark 2.7. We have for the Bernstein–Durrmeyer–Lupaş operators in the case $\alpha = 0$ the *worst case* as seen in Theorem 2.1 and the following last result.

THEOREM 2.8. *Let* $(P_n)_{n\in\mathbb{N}}$ *be defined as in* (1.1), $f \in C(I)$. *Then the estimate in Theorem* 2.4 *does not hold for* $\alpha = 0$.

Remark 2.9. An open question is whether there exists a somewhat weaker estimate than (2.6) for the Bernstein–Durrmeyer–Lupaş operators with $c = c_n = 1$. It seems to be desirable to have an estimate similar as in (Mache and Zhou [20, (3.14)], here with n^{-1}). Clearly, for $f \in C(I)$,

$$\|f - P_n f\|_\infty \leqslant Cn^{-1}\left(\int_{1/\sqrt{n}}^{1/2} \frac{\omega_\varphi^2(f; t)_\infty}{t^2}\frac{dt}{t} + \|f\|_\infty\right). \tag{2.10}$$

Further properties of this approximation processes (also for $c = c_n = [n^\alpha]$, with $0 < \alpha < 1$) will be investigated in a forthcoming publication [18].

2.1. *Some Properties of the Operators* P_n

By simple computations one finds

LEMMA 2.10. *Let* $(P_n)_{n\in\mathbb{N}}$ *be defined as in* (1.1). *Then* $P_n(e_0; x) = 1$ *and*

$$P_n(e_m; x) = \sum_{k=0}^{n} p_{n,k}(x)\,\frac{B(c(n-k)+b+1,\, m+ck+a+1)}{B(ck+a+1,\, c(n-k)+b+1)}, \qquad m \geqslant 1.$$

In particular, we get

$$P_n(e_1; x) = \frac{1}{cn+a+b+2}\,(a+1+cnx)$$

and

$$P_n(e_2; x) = \frac{(a+1)(a+2)+(2ca+3c)\,nx+c^2(n^2x-n(n-1)\,x(1-x))}{(a+b+cn+2)(a+b+cn+3)}.$$

For $\Omega_{r,x} := (t-x)^r$, $r \in \mathbb{N}$, *we obtain*

$$P_n(\Omega_{1,x}; x) = \frac{a+1-(a+b+2)\,x}{cn+a+b+2} \tag{2.11}$$

and

$$P_n(\Omega_{2,x}; x) = \frac{\begin{array}{c} c(c+1)\,nx(1-x)+(a+b+2)(a+b+3)\,x^2 \\ -2x(a+1)(a+b+2)+(a+1)(a+2) \end{array}}{(a+b+cn+2)(a+b+cn+3)}.$$

Moreover, for $x \in I$,

$$P_n(\Omega_{2,x}; x) \leqslant \frac{2(a+1)+(b+1)(b+2)}{(cn)^2} + \frac{(c+1)\,x(1-x)}{cn}\,\psi_n(x), \tag{2.13}$$

where

$$\psi_n(x) := \begin{cases} 1, & x \in E_n \\ 0, & x \in I/E_n \end{cases}; \qquad E_n := \left[\frac{1}{n}; 1-\frac{1}{n}\right].$$

At this point we can note that the order of approximation increases near to the endpoints of the interval I. So let us characterize the local convergence for the positive linear operator P_n by the elements of the Lipschitz class Lip β. Here the local behaviour of a function will be measured by a maximal function $f_\beta^{\sim}$, which is defined by Lenze [11] as

$$f_\beta^{\sim}(x) := \sup_{t \neq x,\, t \in I} \frac{|f(x)-f(t)|}{|x-t|^\beta}, \qquad x \in I, \quad 0 < \beta \leqslant 1. \tag{2.14}$$

We have the following local direct estimate.

THEOREM 2.11. *Let P_n, $n \in \mathbb{N}$, be defined as in* (1.1). *Then for $f \in C(I)$, $x \in I$,*

$$|f(x) - P_n(f; x)| \leqslant f_\beta^{\sim}(x) \left(\frac{(c+1)\, x(1-x)}{cn} + \frac{2(a+1) + (b+1)(b+2)}{(cn)^2} \right)^{\beta/2}, \tag{2.15}$$

where $c := c_n$ is defined in (1.3).

Proof. With the inequality for $\beta \in (0, 1]$

$$|f(t) - f(x)| \leqslant |t-x|^\beta \, f_\beta^{\sim}(x), \qquad (x, t) \in I \times I,$$

we have by using Hölder's inequality for $p = 2/\beta > 1$

$$\begin{aligned} |P_n(f; x) - f(x)| &\leqslant f_\beta^{\sim}(x)\, P_n(|t-x|^\beta; x) \leqslant f_\beta^{\sim}(x) (P_n(|t-x|^{p\beta}; x))^{1/p} \\ &\leqslant f_\beta^{\sim}(x) (P_n(\Omega_{2,x}; x))^{\beta/2}, \end{aligned}$$

which concludes the proof. ▮

Similar results have been proved by Lenze [12], Mache [17], and Mache and Müller [19].

Remark 2.12. It does seem not to be known whether there are local converse theorems. So it is an open problem to give local equivalence results taking into consideration that a simple local change of f can influence the polynomial $P_n f$ on the whole interval I.

In order to prove global direct and converse approximation results we need the following Bernstein type inequalities.

LEMMA 2.13. *Let $\varphi(x) = \sqrt{x(1-x)}$ and $n \in \mathbb{N}$. Then for $f \in C(I)$*

$$\|P_n'' f\|_\infty \leqslant C_1 n^2 \, \|f\|_\infty, \tag{2.16}$$

$$\|\varphi^2 P_n'' f\|_\infty \leqslant C_2 n \, \|f\|_\infty, \tag{2.17}$$

and for smooth functions $f \in C^2(I)$

$$\|P_n'' f\|_\infty \leqslant \|f''\|_\infty, \tag{2.18}$$

$$\|\varphi^2 P_n'' f\|_\infty \leqslant C_3 \, \|\varphi^2 f''\|_\infty, \tag{2.19}$$

where the constants C_1, C_2 and C_3 are independent of n and f.

Proof. For simplicity we consider only the case $a=b=0$. Then $P_n f$ has the representation

$$P_n(f;x)=\sum_{k=0}^{n} p_{n,k}(x)\,\frac{\int_0^1 f(t)\,t^{ck}(1-t)^{c(n-k)}\,dt}{B(ck+1,\,c(n-k)+1)}$$

$$=(cn+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 f(t)\,p_{cn,ck}(t)\,dt. \tag{2.20}$$

Hence for $x\in I$,

$$|P_n(f;x)|\leqslant \|f\|_\infty. \tag{2.21}$$

The first derivative of $P_n f$ is

$$P_n'(f;x)=n(cn+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)\int_0^1 f(t)(p_{cn,c(k+1)}(t)-p_{cn,ck}(t))\,dt$$

$$=n(cn+1)\sum_{k=0}^{n-1} p_{n-1,k}(x)\int_0^1 f(t)\sum_{j=ck}^{c(k+1)-1}(p_{cn,j+1}(t)-p_{cn,j}(t))\,dt$$

$$=n\sum_{k=0}^{n-1} p_{n-1,k}(x)\sum_{j=ck}^{c(k+1)-1}\int_0^1 f'(t)\,p_{cn+1,j+1}(t)\,dt$$

and the second derivative of $P_n f$ is

$$P_n''(f;x)=n(n-1)\sum_{k=0}^{n-2} p_{n-2,k}(x)\left\{\sum_{j=c(k+1)}^{c(k+2)-1}\int_0^1 f'(t)\,p_{cn+1,j+1}(t)\,dt\right.$$

$$\left.-\sum_{j=ck}^{c(k+1)-1}\int_0^1 f'(t)\,p_{cn+1,j+1}(t)\,dt\right\}$$

$$=n(n-1)\sum_{k=0}^{n-2} p_{n-2,k}(x)\sum_{j=ck}^{c(k+1)-1}\sum_{l=0}^{c-1}\frac{1}{cn+2}$$

$$\times\int_0^1 f''(t)\,p_{cn+2,j+1+l+1}(t)\,dt.$$

Making use of

$$\int_0^1 p_{cn+2,j+l+2}(t)\,dt$$

$$=\frac{(cn+2)!}{(j+l+2)!\,(cn-j-l)!}\,B(j+l+3,\,cn-j-l+1)=\frac{1}{cn+3}$$

it follows that

$$\|P''_n f\|_\infty \leqslant \|f''\|_\infty. \tag{2.22}$$

Different representations for $(P_n f)'$ and $(P_n f)''$ are

$$P'_n(f; x) = n(cn+1) \sum_{k=0}^{n-1} p_{n-1,k}(x) \int_0^1 f(t)[p_{cn, c(k+1)}(t) - p_{cn, ck}(t)]\, dt \tag{2.23}$$

and

$$P''_n(f; x) = n(n-1)(cn+1) \sum_{k=0}^{n-2} p_{n-2,k}(x) \int_0^1 f(t)[p_{cn, c(k+2)}(t) - 2p_{cn, c(k+1)}(t) + p_{cn, ck}(t)]\, dt. \tag{2.24}$$

Since for $a \in \mathbb{N}_0$

$$\int_0^1 p_{cn, c(k+a)}(t) = \frac{(cn)!}{(c(k+a))!\,(c(n-k-a))!} B(c(k+a)+1, c(n-k-a)+1) = \frac{1}{cn+1},$$

we obtain from (2.24) the upper bound

$$|P''_n(f; x)| \leqslant 4n(n-1) \sum_{k=0}^{n-2} p_{n-2,k}(x)\, \|f\|_\infty \leqslant C_1 n^2 \|f\|_\infty.$$

In the same way as that in [9, Chap. 9] the other two inequalities are proved. ∎

2.2. *Proof of Theorem* 2.1

We will start to prove the **direct part**.

It is known (cf. Berens and Lorentz [3]) that $f \in \mathrm{Lip}^* \beta$ is equivalent to $f \in \mathrm{Lip}\, \beta$ when $0 < \beta < 1$, and to $f' \in \mathrm{Lip}(\beta - 1)$ when $1 < \beta < 2$. So we obtain a "precise characterization" result for $\beta \neq 1$. For the case $\beta = 1$ the class is $\mathrm{Lip}^*\, 1$ ($\supset$ but $\neq \mathrm{Lip}\, 1$).

At first let $0 < \beta < 1$ for $\alpha = 0$ (Case 1) and $0 < \alpha < 1$ (Case 2):

Then $\omega_1(f; t) \leqslant Ct^\beta$ and, from (2.15),

$$|P_n(f; x) - f(x)| \leqslant C\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\beta/2}.$$

Now let $1<\beta\leqslant\alpha+1<2$. Then $f\in C^1(I)$ and $f'\in \mathrm{Lip}(\beta-1)$, say $\omega_1(f';t)\leqslant Ct^{\beta-1}$. Then

$$\begin{aligned}|P_n(f;x)-f(x)| &\leqslant |P_n(f'(x)\,\Omega_{1,x}+f(t)-f(x)-f'(x)\,\Omega_{1,x};x)|\\ &\leqslant \|f'\|_\infty\,|P_n(\Omega_{1,x};x)|+P_n(C\,|t-x|^\beta;x)\\ &\leqslant C\{\|f'\|_\infty\, n^{-1-\alpha}+(P_n(\Omega_{2,x};x))^{\beta/2}\}\\ &\leqslant C\left\{\|f'\|_\infty\, n^{-1-\alpha}+\left(\frac{x(1-x)}{n}+\frac{1}{n^{2(\alpha+1)}}\right)^{\beta/2}\right\}\\ &\leqslant C\left(\frac{x(1-x)}{n}+\frac{1}{n^2}\right)^{\beta/2}.\end{aligned}$$

The Proof of Theorem 2.1 *for* $\beta=1$ *in the Direct Part.* This part is quite different.

Here we will follow along standard lines using the modified Steklov means defined for $d>0$ by (cf. Butzer and Scherer [5, p. 317], Ditzian and Totik [9], Becker [1, p. 135])

$$f_d(s):=\left(\frac{2}{d}\right)^2\int_0^{d/2}\int_0^{d/2}(2f(s+u+v)-f(s+2u+2v))\,du\,dv. \qquad (2.25)$$

Let $\omega_2(f;t)\leqslant Ct$. We extend f to $[0,2]$ in the same way as Berens and Lorentz [3] such that

$$\omega_2(f;t)^{[0,2]}\leqslant 5\omega_2(f;t)\leqslant 5Ct.$$

For $f\in \mathrm{Lip}^*\,1^{[0,2]}$ we have $f\in \mathrm{Lip}\,\gamma^{[0,2]}$ for any $0<\gamma<1$ [3, p. 694], say

$$\omega_1(f;t)\leqslant Ct^\gamma.$$

For such an extension one obtains

$$f(s)-f_d(s)=\left(\frac{2}{d}\right)^2\int_0^{d/2}\int_0^{d/2}\vec{\Delta}^2_{u+v}f(s)\,du\,dv, \qquad (2.26)$$

$$f'_d(s)=\left(\frac{2}{d}\right)^2\left\{2\int_0^{d/2}\left[f\left(s+u+\frac{d}{2}\right)-f(s+u)\right]du\right.$$
$$\left.-\frac{1}{2}\int_0^{d/2}[f(s+2u+d)-f(s+2u)]\,du\right\}, \qquad (2.27)$$

$$f''_d(s)=\left(\frac{1}{d}\right)^2\{8\vec{\Delta}^2_{d/2}f(s)-\vec{\Delta}^2_d f(s)\}. \qquad (2.28)$$

Here we denote for $\vec{\Delta}_t^2 f$ the second forward differences $\vec{\Delta}_t^2 f(x) := f(x+2t) - 2f(x+t) + f(x)$, and hence

$$\|f - f_d\|_\infty \leqslant \omega_2(f; d) \leqslant Cd, \tag{2.29}$$

$$\|f'_d\|_\infty \leqslant 3\left(\frac{2}{d}\right)\omega_1(f; d) \leqslant Cd^{\gamma-1}, \tag{2.30}$$

$$\|f''_d\|_\infty \leqslant 9d^{-2}\omega_2(f; d) \leqslant Cd^{-1}. \tag{2.31}$$

Applying (2.21), we have for $x \in I$

$$\begin{aligned}|P_n(f; x) - f(x)| &\leqslant 2\,\|f - f_d\|_\infty + |P_n(f_d; x) - f_d(x)| \\ &\leqslant 2\,\|f - f_d\|_\infty + |f'_d(x)\, P_n(\Omega_{1,x}; x)| + \|f''_d\|_\infty\, P_n(\Omega_{2,x}; x) \\ &\leqslant C\left(d + d^{\gamma-1}\frac{1}{n^{1+\alpha}} + \frac{1}{d}\left(\frac{x(1-x)}{n} + \frac{1}{n^{2(1+\alpha)}}\right)\right),\end{aligned}$$

where the constant C is independent of x, d, and n. Choosing now

$$d = \left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{1/2}$$

gives

$$|P_n(f; x) - f(x)| \leqslant C\left(2\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{1/2} + n^{-1}\right) \leqslant C\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{1/2},$$

which completes the proof of the direct part of Theorem 2.1 for $\beta = 1$. ▮

Proof of the Inverse Part. Following the arguments in [3, pp. 694] via the regularization process by Steklov means (see (2.33)), it is sufficient for the converse approximation result to prove (cf. Becker [2])

$$\omega_2(f; h) \leqslant C\{\delta^\beta + h^2\delta^{-2}\omega_2(f; \delta)\}. \tag{2.32}$$

Suppose that

$$|P_n(f; x) - f(x)| \leqslant C\left(\frac{x(1-x)}{n} + \frac{1}{n^2}\right)^{\beta/2},$$

for $0 < \beta < 1 + \alpha$ and $x \in I$.

Let $n\in\mathbb{N}$, $x\in I$, $0<t\leqslant h\leqslant\frac{1}{8}$, $\delta(n,x,t)=\max\{1/n,\varphi(x+t)/\sqrt{n}, \varphi(x-t)/\sqrt{n},\varphi(x)/\sqrt{n}\}$, $x\pm t\in I$.

Then we have

$$\begin{aligned}
&|f(x+t)-2f(x)+f(x-t)|\\
&\quad\leqslant|f(x+t)-P_n(f;x+t)|+2\,|f(x)-P_n(f;x)|\\
&\qquad+|f(x-t)-P_n(f;x-t)|+|\Delta_t^2P_n(f;x)|\\
&\quad\leqslant C(\delta(n,x,t))^\beta+|\Delta_t^2P_n(f-f_\delta;x)|+|\Delta_t^2P_n(f_\delta;x)|.
\end{aligned}$$

We introduce the Steklov means

$$f_\delta(x):=\delta^{-2}\int_{-\delta/2}^{\delta/2}\int_{-\delta/2}^{\delta/2}f(x+u+v)\,du\,dv,\qquad x\in I. \tag{2.33}$$

Here $f_\delta\in C^2(I)$ is taken over $\delta>0$, which will be determined later. We have two well known estimates (see also Becker [2, p. 147]),

$$\|f-f_\delta\|_\infty\leqslant C\omega_2(f;\delta)\qquad\text{and}\qquad\|f''_\delta\|_\infty\leqslant C\delta^{-2}\omega_2(f;\delta).$$

Then with the Bernstein-type inequalities (2.16) and (2.17),

$$\begin{aligned}
&|\Delta_t^2P_n(f-f_\delta;x)|\\
&\quad\leqslant\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}|P''_n(f-f_\delta;x+u+v)|\,du\,dv\\
&\quad\leqslant C\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}\min\left\{n^2,\frac{n}{\varphi^2(x+u+v)}\right\}\|f-f_\delta\|_\infty\,du\,dv\\
&\quad\leqslant C\,\|f-f_\delta\|_\infty\min\left\{n^2t^2,n\frac{6t^2}{\max\{\varphi^2(x),\varphi^2(x+t),\varphi^2(x-t)\}}\right\}\\
&\quad\leqslant 6C\,\|f-f_\delta\|_\infty\,t^2(\delta(n,x,t))^{-2}\\
&\quad\leqslant C\omega_2(f;\delta)(\delta(n,x,t))^{-2}\,t^2.
\end{aligned}$$

Here we have used a known result (see [2, Lemma 2]) that for $0<h\leqslant\frac{1}{8}$,

$$\int_{-t/2}^{t/2}\int_{-t/2}^{t/2}\frac{du\,dv}{\varphi^2(x+u+v)}\leqslant\frac{6t^2}{\max\{\varphi^2(x),\varphi^2(x+t),\varphi^2(x-t)\}}.$$

We also have by (2.18)

$$|\Delta_t^2 P_n(f_\delta; x)| \leqslant \left| \int_{-t/2}^{t/2} \int_{-t/2}^{t/2} P_n''(f_\delta; x+u+v)\, du\, dv \right|$$
$$\leqslant Ct^2 \, \|f_\delta''\|_\infty \leqslant Ct^2\delta^{-2}\omega_2(f; \delta).$$

Combining all of the above estimates, we get

$$|\Delta_t^2 f(x)| \leqslant C\{(\delta(n, x, t))^\beta + \omega_2(f; \delta)\, t^2(\delta(n, x, t))^{-2} + \omega_2(f; \delta)\, t^2\delta^{-2}\}.$$

Putting $\delta = \delta(n, x, t)$ and noting that $0 < t \leqslant h$ we have

$$|\Delta_t^2 f(x)| \leqslant C\{(\delta(n, x, t))^\beta + h^2(\delta(n, x, t))^{-2}\, \omega_2(f; \delta(n, x, t))\}. \quad (2.34)$$

Observing that $\delta(n, x, t)$ is tending to 0 for $n \to \infty$ and that

$$\delta(n, x, t) \leqslant \delta(n-1, x, t) \leqslant 2\delta(n, x, t).$$

For any $\delta > 0$, there exists an $n \in \mathbb{N}$ such that

$$\delta(n, x, t) \leqslant \delta < 2\delta(n, x, t).$$

Therefore for any $\frac{1}{8} \geqslant \delta > 0$ and $0 < t \leqslant h \leqslant \frac{1}{8}$ we have

$$|\Delta_t^2 f(x)| \leqslant C\{\delta^\beta + h^2\delta^{-2}\omega_2(f; \delta)\}, \quad (2.35)$$

where C is independent of h, δ, x and t.

Summarizing, we conclude (2.32) and by using the Berens–Lorentz Lemma [3, pp. 694] we have

$$\omega_2(f; h) = \mathcal{O}(h^\beta), \quad (2.36)$$

hence $f \in \mathrm{Lip}^*\, \beta$, which completes the proof of the first part of Theorem 2.1.

Next we prove the second part. For $\alpha = 0$ and $1 < \beta < 2$ the equivalence of (2.4) and (2.5) does not hold, which can be proved by the same method as in [23]. For $0 < \alpha < 1$ and $1 + \alpha < \beta < 2$ consider $f(x) = x$. It is obvious that (2.5) holds. On the other hand, by taking $x = 1/n$ we can see that

$$|f(x) - P_n(f; x)| = |-P_n(\Omega_{1,x}; x)| \geqslant \frac{a+1-(1/n)(a+b+2)}{cn+a+b+2},$$

showing that (2.4) does not hold for any constant C. Hence (2.4) and (2.5) are not equivalent. The proof of Theorem 2.1 is complete. ▮

The proof of Theorem 2.1 shows that the equivalence can be improved in the following way.

COROLLARY 2.14. *Let* $(P_n)_{n \in \mathbb{N}}$ *be defined as in* (1.1), $f \in C(I)$, *and* $0 < \alpha < 1$. *Then for* $0 < \beta \leqslant \alpha + 1$,

$$f \in \mathrm{Lip}^* \beta \tag{2.37}$$

if and only if

$$|f(x) - P_n(f; x)| \leqslant C \left\{ \left(\frac{x(1-x)}{n} \right)^{\beta/2} + n^{-2(1+\alpha)(\beta/2)} + n^{-1-\alpha} \right\}, \qquad x \in I, \tag{2.38}$$

where the constant C is independent of n and α.

2.3. *Proof of Theorem* 2.4

The essential part for this proof is the equivalence of $\omega_\varphi^2(f; \cdot)_\infty$ and the modified K-functional (cf. [9])

$$\bar{K}_2^\varphi(f; t^2)_\infty := \inf\{ \|f - g\|_\infty + t^2 \|\varphi^2 g''\|_\infty + t^4 \|g''\|_\infty \mid g'' \in C(I)\}, \tag{2.39}$$

i.e.,

$$\omega_\varphi^2(f; t)_\infty \sim \bar{K}_2^\varphi(f; t^2)_\infty.$$

We expand $g \in C^2(I)$ by $g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - v)\, g''(v)\, dv$. Making use of [9, Lemma 9.6.1]) we have for $\varphi^2(x) \geqslant n^{-(1+2\alpha)}$,

$$\begin{aligned}
&|P_n(g; x) - g(x)| \\
&\quad \leqslant |g'(x)|\, |P_n(\Omega_{1,x}; x)| + P_n\left(\left| \int_x^t (t - v)\, g''(v)\, dv \right|; x \right) \\
&\quad \leqslant |g'(x)|\, |P_n(\Omega_{1,x}; x)| + P_n\left(\frac{|t - x|}{\varphi^2(x)} \left| \int_x^t \varphi^2(v)\, |g''(v)|\, dv \right|; x \right) \\
&\quad \leqslant |g'(x)|\, |P_n(\Omega_{1,x}; x)| + P_n\left(\frac{\Omega_{2,x}}{\varphi^2(x)}; x \right) \|\varphi^2 g''\|_\infty \\
&\quad \leqslant C\{ \|g'\|_\infty n^{-1-\alpha} + n^{-1} \|\varphi^2 g''\|_\infty \}.
\end{aligned}$$

For $\varphi^2(x) < n^{-(1+2\alpha)}$ we have

$$\begin{aligned}
|P_n(g; x) - g(x)| &\leqslant C \|g'\|_\infty n^{-1-\alpha} + P_n(\Omega_{2,x}; x) \|g''\|_\infty \\
&\leqslant C \left\{ \|g'\|_\infty n^{-1-\alpha} + \left(\frac{\varphi^2(x)}{n} + \frac{1}{n^{\alpha+1}} \right) \|g''\|_\infty \right\} \\
&\leqslant C\{ n^{-1-\alpha} \|g'\|_\infty + n^{-1-\alpha} \|g''\|_\infty \}.
\end{aligned}$$

Altogether,

$$\|P_n g - g\|_\infty \leqslant C\left\{\frac{1}{n}\|\varphi^2 g''\|_\infty + n^{-1-\alpha}(\|g\|_\infty + \|g''\|_\infty)\right\}.$$

Hence for $\alpha \geqslant 1$

$$\begin{aligned}\|P_n f - f\|_\infty &\leqslant \|P_n(f-g)\|_\infty + \|P_n g - g\|_\infty + \|f-g\|_\infty\\ &\leqslant (2+C)\,\|f-g\|_\infty + C\{n^{-1}\,\|\varphi^2 g''\|_\infty\\ &\quad + n^{-2}\,\|g''\|_\infty\} + Cn^{-1-\alpha}\,\|f\|_\infty,\end{aligned}$$

and by taking the infimum over $g \in C^2(I)$ we obtain

$$\begin{aligned}\|P_n f - f\|_\infty &\leqslant (2+C)\,\bar{K}_2^\varphi\left(f; \frac{1}{n}\right)_\infty + Cn^{-1-\alpha}\,\|f\|_\infty\\ &\leqslant C\left(\omega_\varphi^2\left(f; \frac{1}{\sqrt{n}}\right)_\infty + n^{-1-\alpha}\,\|f\|_\infty\right),\end{aligned}$$

which proves Theorem 2.4.

Using the inequalities in Lemma 2.13, we can prove the inverse part of Theorem 2.6 by the standard method in [9, Chap. 9]. The direct part is a corollary of Theorem 2.4. We omit the details of this proof.

Finally we give the proof of Theorem 2.8.

2.4. *Proof of Theorem* 2.8

Suppose that (2.6) holds for $\alpha = 0$. Take $f(x) = x\ln x - x$, $f \in C(I)$. Then we have

$$\|\varphi^2 f''\|_\infty = \max_{0\leqslant x\leqslant 1}|1-x| \leqslant 1.$$

Hence

$$\|f - P_n f\|_\infty \leqslant C\left(\omega_\varphi^2\left(f; \frac{1}{\sqrt{n}}\right) + n^{-1}\,\|f\|_\infty\right) \leqslant \frac{C}{n}.$$

On the other hand, by Taylor's expansion around x

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)\,f''(u)\,du,$$

we have

$$P_n(f; x) - f(x) = f'(x)\,P_n(\Omega_{1,x}; x) + P_n\left(\int_x^t (t-u)\,f''(u)\,du; x\right).$$

Therefore with the moments of P_n we obtain for $x \in [1/n, 1-1/n]$

$$|f'(x)\, P_n(\Omega_{1,x}; x)| \leqslant \|P_n f - f\|_\infty + P_n\left(\left|\int_x^t (t-u)\, f''(u)\, du\right|; x\right)$$

$$\leqslant \|P_n f - f\|_\infty + P_n\left(\frac{\Omega_{2,x}}{\varphi^2(x)}; x\right) \|\varphi^2 f''\|_\infty$$

$$\leqslant \frac{C}{n} + \frac{C}{\varphi^2(x)}\left(\frac{\varphi^2(x)}{n} + \frac{1}{n^2}\right) \leqslant \frac{4C}{n}.$$

Especially for $x = x_n = 1/n$ we would obtain

$$|f'(x_n)\, P_n(\Omega_{1,x_n}; x_n)| = \left|\ln x_n \frac{a+1-(a+b+2)\, x_n}{cn+a+b+2}\right| \leqslant \frac{4C}{n}.$$

This would mean that for sufficiently large n,

$$\ln n \leqslant C,$$

which is a contradiction. Hence (2.6) cannot hold for $\alpha = 0$, which completes the proof of Theorem 2.8.

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